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ON DEFORMATION OF CONTINUOUS MEDIA IN A WEDGE-LIKE REGION WITH SMOOTH FACES

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Using a quasi-stationary formulation, we investigate a boundary value problem on deformation of isotropic rheological media in a wedge-like region. The deformation of the medium is caused by variation in the angle between the plane sides which together form a plane diffuser, and by the flow rate of mass through this diffuser. The wedge faces are assumed to be perfectly smooth. Notwithstanding the particular properties of the medium, we succeed in determining the displacement field to within a single arbitrary function independent of the polar angle, with all boundary conditions of the problem satisfied. We show that with the quasi-stationary formulation of the problem the partial derivative with respect to time is obtained in terms of the partial derivative with respect to the diffuser angle of opening and the mass flow rate, with both these quantities assumed to be variable. The formula for the partial derivative with respect to time enables us to express any kinematic characteristic (velocity, deformation, rate of deformation, etc.) in terms of the displacements. We successfully integrate the equations of equilibrium and determine the stress field to within a single arbitrary function independent of the polar angle. In this manner we reduce the solution of the boundary value problems on deformation of continuous media in a wedge-like region with smooth faces to determining the dependence of two arbitrary functions on the radius, that is, after substituting the stress and displacement fields obtained into the defining equations of the medium in question.

Some of the problems on deformation of continuous media in wedge-like regions have been solved. Thus we have the Hamel solution [1] of the problem of flow of a viscous fluid through a diffuser, and the Schield solution [2] of the process of extruding a rigid-plastic material through a wedge-like die. Several solutions of the problems on small plane deformations of a nonlinearly elastic wedge are given in the monograph [3]. Exact solutions of the problems of large deformations of an incompressible elastic wedge with arbitrary elastic potential were obtained in [4]. In addition, numerous results of investigations of deformation of continuous media in wedge-like regions appear in [5] and others.

1. Let us consider the kinematics of deformation of a continuous medium in a wedge-like region when the angle of the diffuser opening varies slowly and the mass flow through this diffuser is given. From the condition of smoothness of the diffuser walls it follows that the shear component $E_{r\theta}$ of the Almansi tensor of finite deformations $\mathbf{E}$ [6] and the shear component $\varepsilon_{r\theta}$ of the deformation rate tensor $\boldsymbol{\varepsilon}$ are both zero at the wedge faces. The tensors E_{ij} and ε_{ij} are given in terms of displacements $\mathbf{u}$ and velocity $\mathbf{v}$ by the formulas

$$\mathbf{E} = {}^1/_2(\mathbf{U} + \mathbf{U}^* - \mathbf{U}^*\mathbf{U}), \quad \boldsymbol{\varepsilon} = {}^1/_2(\mathbf{V} + \mathbf{V}^*) \tag{1.1}$$

$$\mathbf{U} = \left\| \begin{matrix} u_{r,r} & \dfrac{u_{r,\theta} - u_\theta}{r} \\ u_{\theta,r} & \dfrac{u_{\theta,\theta} + u_r}{r} \end{matrix} \right\|, \quad \mathbf{V} = \left\| \begin{matrix} v_{r,r} & \dfrac{v_{r,\theta} - v_\theta}{r} \\ v_{\theta,r} & \dfrac{v_{\theta,\theta} + v_r}{r} \end{matrix} \right\|$$

where the asterisk denotes transposition.

Since for isotropic media the problem is symmetrical about the bisectrix of the plane diffuser angle, the components $E_{r\theta}$ and $\varepsilon_{r\theta}$ are equal to zero on this bisectrix. Thus the problem of deformation of a medium in a wedge of apex angle α is equivalent to the same problem for a wedge of apex angle $\alpha/2$. However for a wedge of angle $\alpha/2$ the problem is again symmetric about the bisectrix of the angle $\alpha/2$ and consequently the components $E_{r\theta}$ and $\varepsilon_{r\theta}$ will also be zero on this bisectrix. Continuing this reasoning we conclude that the following relation holds for arbitrary isotropic media in the problem under consideration:

$$2E_{r\theta} = u_{\theta,r} + \frac{1}{r}(u_{r,\theta} - u_\theta) - \frac{1}{r}u_{r,r}(u_{r,\theta} - u_\theta) - \tag{1.2}$$
$$\frac{1}{r}u_{\theta,r}(u_{\theta,\theta} + u_r) = 0, \quad 2\varepsilon_{r\theta} = v_{\theta,r} + \frac{1}{r}(v_{r,\theta} - v_\theta) = 0$$

From the absence of shear in the material it follows that all material planes drawn through the wedge edge are deformed identically, and the deformation field $\mathbf{E}$ as well as the deformation rate $\boldsymbol{\varepsilon}$ are both independent of the polar angle θ, i.e.

$$\mathbf{E} = \mathbf{E}(r), \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(r) \tag{1.3}$$

The properties (1.2) and (1.3) show that the material particles which were situated prior to deformation on the cylindrical surface $r_0 = \text{const}$, will appear after the deformation on a new cylindrical surface $r = \text{const}$, while the material particles lying, prior to deformation, in the plane $\theta_0 = \text{const}$ will lie, after the deformation, in some plane $\theta = \text{const}$. These properties of deformation of material surfaces within the wedge region were postulated in [4], and for the displacements the following expressions with two arbitrary functions $f_1(r)$ and $\varphi(\theta)$ were subsequently derived:

$$u_r = r - \frac{f_1(r)}{\sqrt{1 + \varphi^2(\theta)}}, \quad u_\theta = -\frac{f_1(r)\varphi(\theta)}{\sqrt{1 + \varphi^2(\theta)}} \tag{1.4}$$

The substitution of the displacements from (1.4) into the expression for $E_{r\theta}$ in (1.2), yields an identity, substitution of the displacements into E_{rr} in (1.1) leads to fulfilment of the property (1.3) and substitution into the expression for $E_{\theta\theta}$ produces the following equation for the function $\varphi(\theta)$ provided that $E_{\theta\theta}$ is independent of θ:

$$\frac{d\varphi}{d\theta}\frac{1}{\sqrt{1 + \varphi^2}} = B_1 \tag{1.5}$$

where B_1 is an arbitrary constant. The solution of (1.5) has the form

$$\varphi = \operatorname{tg}(B_1\theta + B_2) \tag{1.6}$$

The nonzero components of the Almansi tensor in (1.1) are expressed in terms of a single arbitrary function $f_1(r)$ by the formulas

$$E_{rr} = \frac{1}{2}\left[1 - \left(\frac{df_1}{dr}\right)^2\right], \quad E_{\theta\theta} = \frac{1}{2} - \frac{1}{2}\left[\frac{f_1(r)(1+B_1)}{r}\right]^2 \tag{1.7}$$

The velocity $\mathbf{v}$ is given in a polar coordinate system in terms of the displacements $\mathbf{u}$ by

$$v_r = \frac{\partial u_r}{\partial t} + v_r\frac{\partial u_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right) \tag{1.8}$$

$$v_\theta = \frac{\partial u_\theta}{\partial t} + v_r\frac{\partial u_\theta}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right)$$

We recall that we carry out this investigation, using a quasi-stationary formulation; in spite of this, we cannot neglect in (1.8) the partial derivatives of displacements with respect to time. Clearly, if the diffuser angle α and the quantity of mass q flowing through the diffuser do not vary with time, then the displacements u_r and u_θ are also invariant and the velocity components v_r and v_θ vanish. Thus u_r and u_θ depend not only on the coordinates r and θ, but also on the parameters α and q, i.e. $\mathbf{u} = \mathbf{u}(r, \theta, \alpha, q)$.

The parameters α and q depend only on time, therefore the displacements $\mathbf{u}$ also depend only on time in accordance with the relations $\alpha(t)$ and $q(t)$. From this it follows that the function f_1 and the quantities B_1 and B_2 also depend on the parameters $\alpha(t)$ and $q(t)$, i.e. $f_1 = f_1(r, \alpha, q)$ and $B_i = B_i(\alpha, q)$, $i = 1, 2$, while the partial derivative with respect to time is expressed in terms of the partial derivatives with respect to the parameters α and q as follows:

$$\frac{\partial}{\partial t} = \omega\frac{\partial}{\partial\alpha} + Q\frac{\partial}{\partial q}, \quad \frac{d\alpha}{dt} = \omega, \quad \frac{dq}{dt} = Q$$

Let the planes of the wedge faces be described by the equations $\theta_1 = 0$ and $\theta_2 = \alpha(t)$. From the equivalence of the deformation of the material planes we have $\theta = \text{const}$, and from the condition that the rate of shear of the material is zero it follows that

$$v_\theta = \frac{\omega}{\alpha}r\theta, \quad v_r = f_2(r) \tag{1.9}$$

where f_2 is an arbitrary function independent of the angle θ. Substituting (1.4) and (1.9) into (1.8), we obtain the following two equations:

$$\omega\frac{\partial f_1}{\partial\alpha} + Q\frac{\partial f_1}{\partial q} + \frac{\partial f_1}{\partial r}f_2 = 0 \tag{1.10}$$

$$\omega f_1\left(\frac{\partial B_1}{\partial\alpha}\theta + \frac{\partial B_2}{\partial\alpha}\right) + Qf_1\left(\frac{\partial B_1}{\partial q}\theta + \frac{\partial B_2}{\partial q}\right) + \frac{\omega\theta}{\alpha}f_1(1+B_1) = 0$$

From the first equation of (1.10) it follows that the function f_2 can be written in the form

$$f_2 = f_{2\alpha}\omega + f_{2q}Q, \quad f_{2\alpha} = -\frac{\partial f_1}{\partial\alpha}\Big/\frac{\partial f_1}{\partial r}, \quad f_{2q} = -\frac{\partial f_1}{\partial q}\Big/\frac{\partial f_1}{\partial r}$$

i.e.

$$v_r = -\left(\omega \frac{\partial f_1}{\partial \alpha} + Q \frac{\partial f_1}{\partial q}\right) \Big/ \frac{\partial f_1}{\partial r} \tag{1.11}$$

The quantities f_1, B_1 and B_2 appearing in the second equation of (1.10) are independent of ω and Q, therefore, equating the coefficients at ω and Q separately to zero, we obtain the following equations:

$$\frac{\partial B_1}{\partial \alpha}\theta + \frac{\partial B_2}{\partial \alpha} + \frac{\theta}{\alpha}(1 + B_1) = 0, \quad \frac{\partial B_1}{\partial q}\theta + \frac{\partial B_2}{\partial q} = 0$$

Since B_1 and B_2 are independent of θ, we obtain

$$\frac{\partial B_1}{\partial \alpha} + \frac{1 + B_1}{\alpha} = 0, \quad \frac{\partial B_2}{\partial q} = \frac{\partial B_2}{\partial \alpha} = \frac{\partial B_1}{\partial q} = 0$$

The solution of these equations with the boundary condition $\varphi(\theta) = 0$ when $\alpha(0) = \alpha_0$, has the form

$$B_1 = \alpha_0 / \alpha - 1, \quad B_2 = k\pi, \quad k = 0, 1, 2, \ldots$$

Finally, for the function $\varphi(\theta)$ we obtain the expression

$$\varphi(\theta) = \operatorname{tg}(\alpha_0 / \alpha - 1)\,\theta\,,$$

Thus the kinematics of deformation of an isotropic continuous medium has been completely determined to within a single arbitrary function $f_1(r, \alpha, q)$, which must be determined after specifying the particular properties of the material. Usually, solving the boundary value problem does not immediately yield the dependence of the function f_1 on the parameters α and q, and in this case the right-hand side of the expression (1.11) for the velocity v_r also becomes indeterminate. In such cases it is more convenient to use (1.9) instead of (1.11) and treat f_2 as the unknown function. The nonzero components of the rate of deformation tensor are expressed in terms of $f_2(r)$ by the formulas

$$\varepsilon_{rr} = \frac{df_2(r)}{dr}, \quad \varepsilon_{\theta\theta} = \frac{f_2(r)}{r} + \frac{\omega}{\alpha}$$

The principal directions of the tensors E and ε coincide with the directions of the axes of the polar coordinate system. Using this we write the expression for the medium density in terms of the principal values of the Almansi tensor in the Lagrangian form [6]

$$\rho = \rho_0 \sqrt{(1 - 2E_{rr})(1 - 2E_{\theta\theta})}$$

where ρ_0 is the initial density prior to deformation. Substituting (1.7) into this expression we obtain

$$\rho = \frac{\alpha_0 \rho_0}{\alpha r}\left| f_1 \frac{df_1}{dr} \right| \tag{1.12}$$

Substitution of the expressions (1.9), (1.11) and (1.12) into the equation of continuity yields an identity. It remains to consider the law of conservation of the mass flowing through the diffuser, which plays the part of a boundary condition. In the problems of pressing a mass through a plane diffuser of constant angle, this law is usually written in the form

$$Q = \int_0^\alpha \rho v_r r\, d\theta \tag{1.13}$$

In the present case the diffuser angle α varies with time, and this must be accounted for in (1.13). During the time Δt the diffuser angle changes by $\Delta\alpha$. Part of the mass

$\Delta q = Q\Delta t$ of medium which enters the diffuser, passes through the surface $r = \text{const}$ and the remainder fills the sector of central angle $\Delta\alpha$ and radius r, i.e.

$$Q\Delta t = \Delta t \int_0^\alpha \rho v_r r d\theta + \Delta\alpha \int_0^r \rho r dr \tag{1.14}$$

At the limit $\Delta t \to 0$ (1.14) yields the required law of conservation of the mass consumed:

$$Q = \int_0^\alpha \rho v_r r d\theta + \omega \int_0^r \rho r dr \tag{1.15}$$

Using (1.11) and (1.12) we can reduce (1.15) to the form

$$-Q = \alpha_0\rho_0\beta f_1 \left(\omega \frac{\partial f_1}{\partial \alpha} + Q \frac{\partial f_1}{\partial q}\right) + \alpha_0\rho_0\omega \frac{\beta}{2\alpha} [f_1^2 - f_1^2(0)] \tag{1.16}$$

$$\beta = \operatorname{sign}\left(f_1 \frac{df_1}{dr}\right)$$

Repeating the procedure used with (1.10) we equate the coefficients of Q and ω in (1.16) separately to zero, and obtain the following two equations:

$$2\alpha f_1 \frac{\partial f_1}{\partial \alpha} = f_1^2(r) - f_1^2(0), \quad \alpha_0\rho_0 f_1 \frac{\partial f_1}{\partial q} = -\beta \tag{1.17}$$

the second of which yields

$$f_1^2 = C(\alpha, r) - \frac{2\beta}{\alpha_0\rho_0} q \tag{1.18}$$

where $C(\alpha, r)$ is an arbitrary function. The substitution of (1.18) into the first equation of (1.17) yields the following equation for the function $C(\alpha, r)$:

$$\alpha \partial C / \partial \alpha = C(\alpha, r) - C(\alpha, 0) \tag{1.19}$$

If no mass is consumed $(q = 0)$, then the material particles at the wedge edge remain stationary when the wedge faces close. This means that the displacements are zero when $r = q = 0$

$$f_1(r, \alpha, q) = 0 \quad \text{for} \quad r = q = 0 \tag{1.20}$$

Setting $r = q = 0$, in (1.18), we find

$$C(\alpha, 0) = 0 \tag{1.21}$$

The solution of (1.19) with condition (1.21), has the form

$$C(\alpha, r) = \beta C_1(r)\alpha, \quad C_1(0) = 0 \tag{1.22}$$

where $C_1(r)$ is an arbitrary function. Substituting (1.22) into (1.18), we arrive at the following expression for $f_1^2(r, \alpha, q)$:

$$f_1^2(r, \alpha, q) = \beta\left[\alpha C_1(r) - \frac{2q}{\alpha_0\rho_0}\right] \tag{1.23}$$

Next we establish an important property of the function $C_1(r)$. Let us differentiate (1.23) with respect to r

$$2f_1 \partial f_1 / \partial r = \beta\alpha dC_1 / dr \tag{1.24}$$

The sign of the left-hand side of (1.24) is the same as that of β, defined in (1.17), therefore from (1.24) it follows that

$$dC_1 / dr \geqslant 0, \quad C_1(r) \geqslant 0, \quad C_1(0) = 0 \tag{1.25}$$

i.e. $C_1(r)$ is a positive monotonously increasing function. The value $r = r^*$ at which the right-hand side of (1.23) becomes zero

$$C_1(r^*) = 2q / (\alpha_0 \alpha \rho_0) \qquad (1.26)$$

By virtue of (1.25), Eq. (1.26) has a unique solution only when $q > 0$, i.e. when the mass of the medium enters the diffuser through an infinitely narrow slit in its edge. When the process is reversed, the mass is squeezed out of the diffuser and $q < 0$, Eq. (1.26) has no solution. The radius r^* separates the mass which was present in the diffuser at the start of the squeezing process from the mass fed into the diffuser. This follows from the fact that $u_r = r$, $u_\theta = 0$ when $r = r^*$ and also from the identity

$$q \equiv \alpha \int_0^{r^*} \rho r dr$$

The condition that both sides of (1.23) are positive implies that the sign of β is the same as that of the expression within the square brackets in (1.23). By virtue of the monotonous character of the function $C_1(r)$ we have the following relations:

for $q > 0$, $\beta = 1$ when $r > r^*$ and $\beta = -1$ when $r \leqslant r^*$; for $q \leqslant 0$, $\beta = 1$

Substituting (1.23) into (1.4), (1.7) and (1.11) and into the expressions for ε_{rr} and $\varepsilon_{\theta\theta}$, we obtain the following expressions for the displacements, the velocities and the principal values of tensors of the deformation and rate of deformation:

$$u_r = r - \sqrt{B} \cos\left(\frac{\alpha_0}{\alpha} - 1\right)\theta, \quad u_\theta = -\sqrt{B} \sin\left(\frac{\alpha_0}{\alpha} - 1\right)\theta \qquad (1.27)$$

$$B = \beta\left[\alpha C_1(r) - \frac{2q}{\alpha_0 \rho_0}\right]$$

$$v_r = \frac{2Q}{\alpha_0 \alpha \rho_0 C_1'(r)} - \frac{\omega C_1(r)}{\alpha C_1'(r)}, \quad v_\theta = \frac{\omega}{\alpha} r\theta$$

$$E_{rr} = \frac{1}{2}\left(1 - \frac{\alpha^2 C_1'^2}{4B}\right), \quad E_{\theta\theta} = \frac{1}{2} - \frac{B\alpha_0^2}{2\alpha^2 r^2}$$

$$\varepsilon_{rr} = \frac{\omega (C_1 C_1'' - C_1'^2)}{\alpha C_1'^2} - \frac{2QC_1''}{\alpha_0 \alpha \rho_0 C_1'^2}, \quad \varepsilon_{\theta\theta} = \frac{\omega}{\alpha} - \frac{\omega C_1}{\alpha r C_1'} + \frac{2Q}{\alpha_0 \alpha \rho_0 r C_1'}$$

From (1.27) we see that $E_{\theta\theta} \to 1/2$ and $E_{rr} \to -\infty$ when $r \to r^*$. This means that on approaching the surface $r = r^*$ the material elements begin to experience an unlimited compression in the radial direction, and an unlimited stretching in the circumferential direction. We also note that when the product $\omega Q > 0$, a radius r^{**} exists which separates two mutually opposing flows. From the condition $v_r = 0$ we find the radius r^{**} as the root of the equation $C_1(r^{**}) = 2Q / (\alpha_0 \omega \rho_0)$.

For an incompressible continuous medium we obtain. after substituting the condition $\rho = \rho_0$ into (1.12), $C_1(r) = r^2 / \alpha_0$ and find, that in this case the deformation and the rate of deformation become constant over the whole wedge region when $q = Q = 0$.

2. Let us now investigate the stresses in the problem under consideration. We assume that the shear and rate of shear are zero everywhere within the region of the wedge. It is therefore natural to assume that in this case the shear stress τ in isotropic continuous media will be zero: $\tau = 0$. We shall also assume that by virtue of the equivalence of the deformation of all material planes passing through the wedge edge, the stresses σ are, similarly to the tensors $\mathbf{E}$ and ε, independent of the angle θ : $\sigma = \sigma(r)$. In this case we can integrate the equations of equilibrium to obtain

$$\sigma_{rr} = \frac{f_3(r)}{r}, \quad \sigma_{\theta\theta} = \frac{df_3(r)}{dr}, \quad \tau = 0 \tag{2.1}$$

where $f_3(r)$ is an arbitrary function of the parameters α, q of the problem, of the derivatives with respect to time, and of the physical constants that describe the material properties; this function depends on the radius r only.

The obtained expressions (1.27) and (2.1) simplify considerably the process of solving the boundary value problem of pressing a continuous medium through a plane diffuser with a variable angle.

Let us write the defining equations for an isotropic continuous medium in the form

$$F_{kl}(\sigma_{ij}, E_{ij}) = 0 \tag{2.2}$$

where F_{kl} is an integro-differential tensor operator. Subsituting the corresponding expressions from (1.27) and (2.1) into (2.2) with $k = l = r$ and $k = l = \theta$, we obtain two integro-differential equations. Since the relations (1.27) and (2.1) ensure a priori that the boundary conditions, the equations of equilibrium and equations of conservation of mass all hold, therefore the solution of these two integro-differential equations for the two unknown functions $C_1(r)$ and $f_3(r)$ represents the required solution of the boundary value problem in question.

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